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The effects of intense sound levels and steady flow on the propagation of plane-waves in air

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THE EFFECTS OF INTENSE SOUND LEVELS AND STEADY FLOW ON THE PROPAGATION OF PLANE-WAVES IN AIR

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ABSTRACT

This thesis is a theoretical investigation of the effects of intense sound levels and steady flow on the propagation of plane-waves in tubes of constant diameter. The analysis is limited to a study of the change of wave profile due to the intense sound levels, and steady flow. However, corrections for effects of viscous and heat conduction losses at tube walls are also presented.

To simplify the analysis, the problem is treated by considering the steady flow and finite-amplitude-sound wave effects separately. For finite-amplitude sound only, two approaches are evaluated. The first uses small perturbations to derive a finite-amplitude wave equation. It is assumed that a velocity potential exists, and that the relationship between acoustic variables in the linear equation also applies to second-order terms. Because of the nature of the nonlinearity of the finite-amplitude wave equation, only an approximate solution is obtained; the solution is, therefore, limited to weak nonlinearities. A second approach, solution by the method of characteristics, is also investigated.

Trimmer's investigation of steady flow is presented and discussed. His approach for small amplitude sound waves is extended to the approximate solution obtained for finite amplitude waves. A final correction to account for tube wall losses is also described and presented.

Results obtained show that for sound pressure levels up

to 140 dB in many practical systems finite amplitude effects are relatively small, but may become significant in the presence of steady flow; tube attenuation reduces finite-amplitude effects, but not significantly for frequencies less than 1000 Hz in tubes greater than 1 in. diameter and less than 5 meters in length.

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NOMENCLATURE

a	=	acceleration
a_x	=	acceleration in x-direction
A	=	constant particle displacement
b	=	constant acoustic particle velocity
c	=	sound velocity
c_o	=	sound velocity at the equilibrium state
dB	=	unit of sound pressure level (SPL)
D	=	constant sound pressure
f	=	frequency in Hz
h	=	distortion coefficient
h_e	=	distortion coefficient in Eulerian description
h_L	=	distortion coefficient in Lagrangian description
k	=	wave number = ω/c
k_o	=	wave number = ω/c_o
p	=	sound or acoustic pressure
p_{dc}	=	$P_o + P_{r_L}$ = constant pressure term
p_{re}	=	Rayleigh radiation pressure
p_{r_L}	=	Langevin radiation pressure
p_1	=	sound pressure satisfying the linear wave equation
p_2	=	correction term of sound pressure
p_{1max}	=	maximum magnitude of p_1
p_2'	=	second harmonic sound pressure
p'_{2max}	=	maximum magnitude of p_2'
P	=	total or instantaneous pressure = $p + P_o$
P_o	=	equilibrium or static pressure

s	=	condensation = $(\rho - \rho_o)/\rho_o$
t	=	time
u,v,w,	=	particle velocity in the x,y,z directions respectively
u_1	=	acoustic particle velocity satisfying the linear wave equation
u_2	=	correction term of acoustic particle velocity
V	=	steady flow velocity
x,y,z,	=	Catesian coordinates of a fluid particle
X,Y,Z,	=	particle displacement in the x,y,z, directions respectively
α,β	=	characteristic equation
γ	=	ratio of specific heats
δ	=	incremental or acoustic density = $\rho-\rho_o$
λ	=	wave length
ξ	=	particle displacement
ρ	=	instantaneous mass density
ρ_o	=	equilibrium or static mass density
ρ_1	=	acoustic mass density satisfying the linear wave equation
ρ_2	=	correction term of acoustic mass density
ψ	=	velocity potential
ψ_1	=	velocity potential satisfying the linear wave equation
ψ_2	=	correction term of velocity potential
ω	=	circular frequency = $2\pi f$

I. INTRODUCTION

This thesis is part of a series of researches following the work done by Buckley (1)* and Simon (2) on the analysis of wave propagation as related to the evaluation of the performance of acoustic filters.

Buckley (1) discusses the design, construction, and use of a standing wave tube for the analysis of sound-attenuation devices. He describes the measurement of tube attenuation, the reflection and transmission characteristics of small expansion chambers, and the performance of the anechoic termination device.

Simon (2) redesigned the standing wave tube to improve its accuracy and versatility, and measured reflection and transmission characteristics of seven reactive filter elements. All calculations of the reflection and transmission characteristics were performed by a digital computer.

Buckley's and Simon's efforts were primarily experimental. This thesis is primarily theoretical in nature. Finite-amplitude waves and the steady flow conditions are considered, an ideal gas is assumed, and the effects of viscosity and heat conduction are neglected. Plane waves are assumed. Experimental measurements on the characteristics of the expansion chamber and the effect of tubing length could be made to verify the theoretical results obtained here by modifying the test equipment used by Simon.

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* Number in parenthesis represents order of bibliography

In the analysis of the finite-amplitude wave propagation phenomenon, only second-order terms based on the plane-wave theory are considered, and it is assumed that the wave is continuous and different from a shock wave. During the analysis it is assumed that it does not deviate very much from the infinitesimal-amplitude wave theory. The reason for this is the desire to use elementary plane-wave theory as a stepping stone to solve the finite-amplitude wave problem.

As Rayleigh (3) has pointed out:

> "In some parts of the subject, all that we can do is to solve those problems whose mathematical conditions are sufficiently simple to admit of solution, and to trust to them and to general principles not to leave us quite in the dark with respect to other questions in which we may be interested."

II. REVIEW OF LITERATURE

Acoustic phenomena are generally described by an infinitesimal (or linear) theory. In other words, we consider

(a) $u << c_0$

(b) $\xi << \lambda$

(c) $p << P_0$

(d) $\delta << \rho_0$

The intensity of sounds normally produced in solids, liquids, and gases is such as to make this a reasonable approximation.

There are, however, several elastic-wave phenomena which are not adequately described by an infinitesimal theory. One of these is "streaming". It was observed by Meissner (4) that a vortical flow of air took place in the vicinity of a quartz plate when it was driven at high amplitude. This streaming was called "quartz wind". Another is the distortion of a wave form of a high-intensity sound as it is propagated through a fluid. We shall now investigate the propagation of such high-intensity or finite-amplitude waves. It then becomes necessary to reconsider the mathematical simplifications used in deriving the elementary wave equation to establish an equation wherein higher-order terms be taken into account.

The characteristic features of finite-amplitude waves will be discussed in detail through the example of a plane wave. The plane wave is the simplest type of wave motion propagated through a fluid medium: the characteristic properties of all points on any given plane perpendicular to the direction of wave propagation are the same; and, because the analysis is one-dimensional, the physical phenomena of wave motion and the resulting mathematical analysis are simpler.

The equation of one-dimensional plane waves of finite amplitude without dissipation was first examined mathematically by Poisson (5) in 1808. He began with the momentum equation and equation of continuity in Eulerian coordinates. Under the assumption of Boyle's law ($P = c^2\rho$), he proved that for waves traveling in one direction (positive) the circumstances of the propagation are expressed by

$$u = f\{x-(c+u)t\} \qquad (1)$$

in which f denotes an arbitrary function. When u can be neglected in comparison with c, this equation reduces to the familiar equation applicable to infinitesimal waves.

The meaning of Eq. (1) is that, in general, an acoustic disturbance advances with a velocity equal not to c, but to c+u. However, from elementary theory, we know that an infinitesimally small disturbance (in which $u \ll c$) is propagated with a certain velocity c, relative to the parts of the medium undisturbed by the wave.

A closer discussion of the solution represented by Poisson's integral was given by Stokes (6) who pointed out the difficulty which ultimately arises from the motion becoming discontinuous. When a pressure discontinuity occurs, a condition exists for which the usual differential equations are not applicable, and the subsequent progress of the motion in a lossless fluid has not been determined. It is possible, as suggested by Stokes, that some sort of reflection ensues.

The assumption that in a progressive wave there is a definite relation between u and ρ form the basis of Earnshaw's investigation (7). Earnshaw worked with the Lagrangian-form of the equation in which the motions of particular particles are followed, and he obtained complete solutions for a wave progressing in one direction.

Both Poisson's and Earnshaw's solutions can be used to calculate the time or distance traveled by a finite-amplitude progressive wave before a discontinuity develops.

Riemann's (8) equations are more general than anything previously given, as they are not limited to a single progressive wave. He introduced the two Riemann variables and employed Eulerian coordinates for his equations.

Rayleigh (9) gave a brief history on the study of one-dimensional finite-amplitude wave with and without dissipation.

The explicit solution for u was carried out by Fubini (10) in 1935. The solution in Lagrangian form is an approximation which is justified by the smallness of the Mach number.

In the analysis of the phenomenon of finite amplitude waves, only implicit solutions are available. These implicit solutions could be used to understand more about finite-amplitude wave phenomena, but it is difficult to apply them in a practical way such that an experiment could be designed to justify the theory. So the search for an explicit solution is the aim of this thesis.

A. The Difference Between the Eulerian and Lagrangian Descriptions.

In deriving the equation of motion for sound waves propagating in a fluid, there are two descriptions commonly employed to describe the motion. One of them identifies each particle* of the fluid by giving its position at t=0, with subsequent positions described by a function of time and the initial position. Thus, it actually describes the motion of specific fluid particles. This description is called the "Lagrangian description".

The advantage of the Lagrangian description is that it provides good insight into what is going on inside the fluid during the flow process. For example, $X(x_o,t)$ represents the displacement of particles in a fluid expressed in Langrangian description, where motion is restricted to the x dimension. We want to know what happens to the displacement of a certain fluid element as time goes on. Here x_o is defined as the position of each particle at a specified time, for example, when t=0. If a certain particle under consideration is at

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* The term "particle" here refers to a small element of a continuous medium throughout which all properties are assumed to be uniform.

x_o when t=0, then $X(x_o,t)$ becomes the displacement history of that very particle. The displacement of this particle at any subsequent time is found by substituting the appropriate value for t. The velocity of this particle at time t is simply $(dX/dt)x=x_o$. If dX/dt is greater than zero, it means that the particle displacement is increasing. The same is true of the density ρ; if the time rate of change $d\rho/dt$ is less than zero, it means the density of the particle under consideration is decreasing as time goes on (the volume of the particle is expanding).

However, the Lagrangian description also has its disadvantages. Since the spatial coordinate system moves with the fluid, it is difficult to determine what the fluid is doing at some specified point in space at a given time.

An alternative notation, the "Eulerian description", sets up a coordinate system fixed in space and describes the properties of a fluid particle which happens to be at a given point at a given time. Therefore, the various quantities measured at a fixed point in space at a given time, and therefore, correspond to different portions of the fluid as the fluid streams past the point.

To find the total derivative, df/dt, of some property f of the fluid, a specific particle in the fluid is being considered.

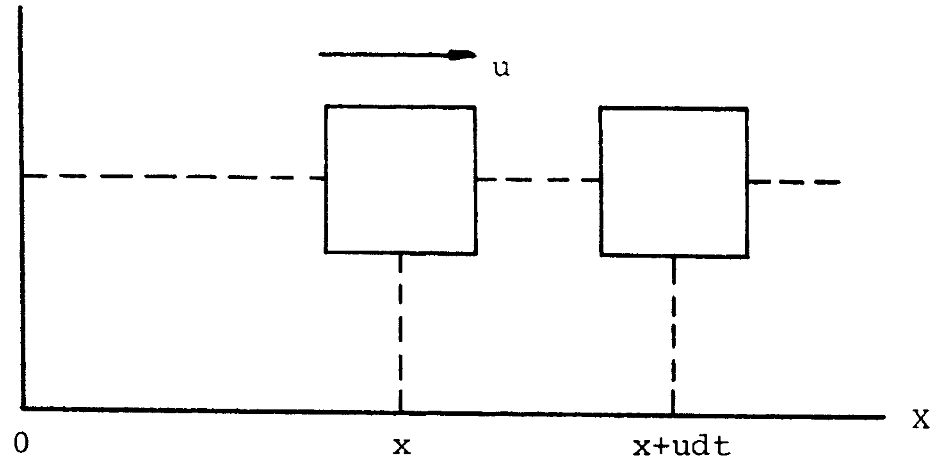

Fig. 1 One-dimensional flow of a fluid particle.

For simplicity a one-dimensional case is shown in Fig. 1. It is desired to find out what is the rate of change of property f in the fluid particle. Assume that in the infinitesimal time period dt, the particle originally at x moved to x+udt, where u is the velocity of the particle. From the definition of the Eulerian description, the fluid-property change of the particle in the time period dt is

$$df= (df/dt)dt=f(x+udt,t+dt)-f(x,t)$$

By Taylor's expansion and neglecting higher order terms

$$f(x+udt,t+dt)=f(x,t)+(\partial f/\partial x)udt+(\partial f/\partial t)dt$$

$$df/dt=\partial f/\partial t+u(\partial f/\partial x)$$

Thus, the true acceleration of the fluid particle at x,t is given in terms of the Eulerian velocity function u(x,t) by

$$du/dt=(\partial u/\partial t)+u(\partial u/\partial x)$$

The above equation illustrates the difference between the Lagrangian and Eulerian descriptions. The first term on the right is the local variation, and the second term describes the spatial variation of the particle velocity.

The Lagrangian formulation describes a given particle's displacement and other properties as functions of time and the particle's initial position, while the Eulerian formulation describes the properties of a medium at a given location and time. Unless it is desired to know the exact path of a given particle, the Eulerian description will be more useful.

The crucial difference between Lagrangian and Eulerian description is in the time rate of change. In the Lagrangian description, the total derivative df/dt of some property f of the fluid is the time rate of change of f associated with a given fluid particle. In the Eulerian description, the partial derivative $\partial f/\partial t$ is the change of f at a fixed point x as the fluid streams past. So, if the velocity is expressed in the Lagrangian description by $u(x_o,t)$, the total derivative $(du/dt)_{x=x_o}$ is the history of the acceleration of the particle located at x_o when $t=0$. If the velocity is expressed in the Eulerian description by $u(x,t)$, then the true acceleration of a particle at x,t is $a=du/dt=(\partial u/\partial t)+u(x,t)\cdot(\partial u/\partial x)$. However, it may be difficult to determine which particle occupies a fixed point in space x at time t.

For the ordinary sound waves that are experienced daily, the excess pressure produced by the acoustic disturbance is below 280 newtons/square meter corresponding to a sound pressure level 140 db ref. 0.00002 newton/square meter and is small compared with the ordinary atmospheric pressure 101,000 newton/square meter; the corresponding acoustic particle velocity* u is also small. Since $\partial u/\partial t$ and $\partial u/\partial x$ are of the same order, the product $u(\partial u/\partial x)$ can be neglected compared with the term $\partial u/\partial t$. The first-order equation of acoustic wave motion is satisfactory to explain most acoustic phenomena and in this case there is no difference between the Lagrangian and Eulerian descriptions. But if the acoustic wave is finite, then the particle velocity will no longer be small; also, under the condition where the medium is moving as a whole, then the particle velocity may not be small even for infinitesimal amplitude wave motion, as shown in Trimmer's investigation (20). For this reason, second order terms must be considered, and the differences between the Lagrangian and Eulerian description can no longer be neglected.

The reason for the preference of the Eulerian description is as follows: In the experiments following the theoretical analysis, if the Lagrangian description is being used, then the coordinate system moves with the fluid particle, and it is not easy to determine which particle actually occupies a specific point in space at a given time. However, the Eulerian description offers a convenient method for specifying the properties such as pressure, and particle velocity as functions of spatial coordinates and time. For example, if it is desired to measure the pressure in the

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* The velocity of a particle relative to an observer at rest.

medium, it is impossible to follow a specific particle while it moves along because of the physical difficulty encountered. For this reason, predictions according to the Lagrangian description are difficult to compare with measured results. In contrast, it is easy to set the pressure transducer at a specific point in space and then compare the result with the Eulerian description.

B. Finite-Amplitude Wave Equation - Lagrangian Description

The derivation of the exact differential equation for sound wave propagation in air involves the continuity equation, Newton's force equation and the equation expressing the relation between pressure and specific volume in a gas.

Here the medium is at rest and in thermodynamic equilibrium. The Lagrangian description is being used and there is one-dimensional motion only.

The following mathematical derivation of the wave equation is an excerpt from Thuras, Jenkins and O'Neil's work (16).

Following Rayleigh (11), let y and $y+(\partial y/\partial x)\,dx$ be the actual distances at time t from the plane $x=0$ to neighboring layers of air whose undisturbed positions are defined by x and $x+dx$, respectively, in Fig. 2. The displacement is thus $\xi=y-x$, and the equation of continuity of the fluid is

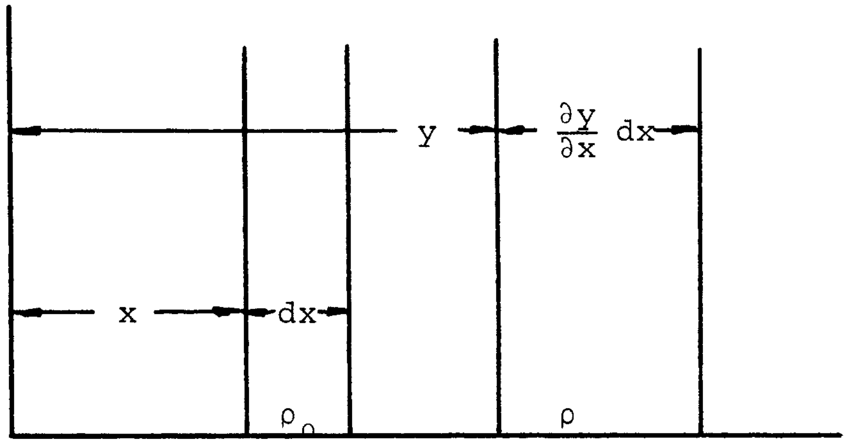

Fig. 2. One-dimensional motion of a fluid particle.

$$\rho\left(\frac{\partial y}{\partial x}\,dx\right)=\rho_o dx$$

or

$$\rho=\rho_o\left(\frac{\partial y}{\partial x}\right)^{-1}=\rho_o\left(1+\frac{\partial \xi}{\partial x}\right)^{-1} \qquad (2\text{-}1)$$

where ρ and ρ_o are the densities of the fluid in the disturbed and undisturbed states, respectively. If the effect of viscosity is neglected the equation of motion for the element of mass $\rho(\frac{\partial y}{\partial x})dx$ is

$$\frac{\partial^2 y}{\partial t^2}\left(\rho\ \frac{\partial y}{\partial x}\ dx\right)=\frac{\partial^2 y}{\partial t^2}(\rho_o dx)=-\ \frac{\partial P}{\partial y}\ \frac{\partial y}{\partial x}\ dx$$

$$\rho_o\left(\frac{\partial^2 \xi_2}{\partial\ t}\right)\ =\ -\left(\frac{\partial P}{\partial x}\right) \qquad (2\text{-}2)$$

Note that P is the pressure at the point y which moves with the air particle, not the pressure at fixed point.

From Eq. (2-1), differentiating both sides with respect to x,

$$\frac{\partial \rho}{\partial x}\ =\ -\ \rho_o\left(1+\ \frac{\partial \xi}{\partial x}\right)^{-2}\ \frac{\partial^2 \xi}{\partial x^2}$$

$$\rho_o{}^{-1}=\ -\left(1+\ \frac{\partial \xi}{\partial x}\right)^{-2}\left(\frac{\partial^2 \xi}{\partial x^2}\right)\ \left(\frac{\partial \rho}{\partial x}\right)^{-1} \qquad (2\text{-}3)$$

Substituting Eq. (2-3) into Eq. (2-2)

$$\frac{\partial^2 \xi}{\partial t^2} = -(\rho_o)^{-1}(\frac{\partial P}{\partial x}) = (1+ \frac{\partial \xi}{\partial x})^{-2} \frac{\partial^2 \xi}{\partial x^2} \frac{\partial P}{\partial x} \frac{\partial x}{\partial \rho} \tag{2-4}$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial P}{\partial \rho} (1+ \frac{\partial \xi}{\partial x})^{-2} \frac{\partial^2 \xi}{\partial x^2}$$

By virtue of Eq. (2-1), Eq. (2-2) is linear in ξ only if $\frac{\partial P}{\partial \rho} = K\rho^{-2}$ or $\frac{\partial P}{\partial v} = -K$, where $v = \frac{1}{\rho}$ and K is a constant. Eq. (2-4) could be written as

$$\frac{\partial^2 \xi}{\partial t^2} = \{K\rho_o^{-2}(1+ \frac{\partial \xi}{\partial x})^2\} (1+ \frac{\partial \xi}{\partial x})^{-2} \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial t^2} = K\rho_o^{-2} \frac{\partial^2 \xi}{\partial x^2}$$

which is linear in ξ. This condition is not satisfied during the finite variations of state of a gas, but is approximately satisfied when the variations are very small. For isothermal changes the relation is $Pv = p_o v_o$ and for adiabatic changes

$$\frac{P}{P_o} = (\frac{v_o}{v})^{\gamma} = (\frac{\rho}{\rho_o})^{\gamma} \tag{2-5}$$

where γ is the ratio of the specific heats and P_o, v_o are the undisturbed equilibrium pressure and specific volume, respectively. In either case, for very small variations, the Pv curve is practically constant. From Eqs. (2-1),(2-4), (2-5), the exact equation of adiabatic plane wave motion in a nonviscous fluid is

$$\frac{\partial P}{\partial \rho} = \frac{P_o \gamma}{\rho_o} (\frac{\rho}{\rho_o})^{\gamma-1}$$

$$\frac{\partial^2 \xi}{\partial t^2} = c_o^2 (1+ \frac{\partial \xi}{\partial x})^{-(\gamma+1)} \frac{\partial^2 \xi}{\partial x^2} \tag{2-6}$$

where $c_o^2 = \frac{\gamma P_o}{\rho_o}$. This is the exact equation according to the Lagrangian description.

It will be seen that for small values of $\frac{\partial \xi}{\partial x}$ this equation approximates the wave equation $\frac{\partial^2 \xi}{\partial t^2} = c_o^2 \frac{\partial^2 \xi}{\partial x^2}$ for small amplitude waves traveling with a velocity c_o^2. The value of the velocity c for a given value of a condensation s relative to the undisturbed medium, as given by Eq. (2-6), may therefore be written as follows.

From the definition of condensation

$$s = \frac{\rho - \rho_o}{\rho_o} = \frac{\rho}{\rho_o} - 1 = (1 + \frac{\partial \xi}{\partial x})^{-1} - 1$$

$$(1+s) = (1 + \frac{\partial \xi}{\partial x})^{-1}$$

Comparing Eq. (2-6) with $\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$

$$c^2 = c_o^2 (1 + \frac{\partial \xi}{\partial x})^{-(\gamma+1)}$$

$$c = c_o (1+s)^{\frac{(\gamma+1)}{2}} \qquad (2\text{-}7)$$

From Eq. (2-7), the velocity of propagation increases with increasing density. Assume there is a propagating sinusoidal sine wave; then the crest of the wave will gradually approach the trough. Such a process would ultimately lead to a discontinuous wave front beyond which the analysis has no meaning. As Rayleigh (12) points out, this tendency is held in check by the divergence of the wave and the influence of attenuation (viscosity and heat conduction), both of which tend to reduce the condensation, and consequently the velocity, to normal or to small amplitude values.

C. Infinitesimal-Amplitude Wave Equation - Eulerian Description

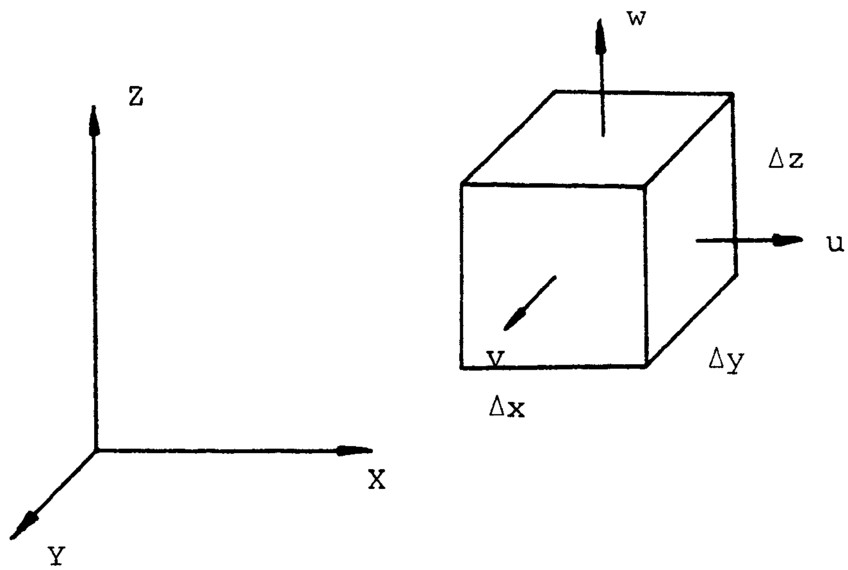

Figure 3. A volume element in space.

Considering a small volume element of the fluid medium as shown in Fig. 3, the difference between the efflux and influx of the medium in this element is equal to the time rate of growth of mass in the element by the principle of continuity.

The simplest method for obtaining the mathematical expression of the principle of continuity is by consideration of the elemental parallelepiped of dimensions $\Delta x, \Delta y, \Delta z$, as shown in Fig. 3. By considering the influx and efflux through each pair of faces respectively, the difference between the latter and the former for the parallelepiped is found to be

$$- \left\{ \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right\} \Delta x \Delta y \Delta z \qquad (2\text{-}8)$$

The rate of growth of mass in the parallelepiped is $\frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z$. Equating these two quantities, the equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \qquad (2\text{-}9)$$

Making the substitution $\rho = \rho_o(1+s)$, Eq. (2-9) becomes

$$\rho_o \frac{\partial s}{\partial t} + \rho_o(1+s) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \rho_o\left(u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z}\right) = 0 \qquad (2\text{-}10)$$

Now in acoustics, the condensation is a very small quantity compared with unity when considering an infinitesimal disturbance. Therefore, terms like $s(\partial u/\partial x)$ and $u(\partial s/\partial x)$ in comparison with $\partial u/\partial x$ can be neglected, and to this approximation the continuity equation becomes

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2\text{-}11)$$

From definition of velocity potential

$$u = -\frac{\partial \psi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial y}, \quad w = -\frac{\partial \psi}{\partial z}$$

Eq. (2-11) becomes

$$\frac{\partial s}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad (2\text{-}12)$$

in a more compact form.

To replace Eq. (2-12) by an equation in which ψ is the only dependent variable, the hydrodynamical equation of motion can be used.

Considering the same elemental parallelepiped in Fig. 3, the equation of motion for the x-component is derived in detail. The other two components may be obtained in a similar manner. Assume there are no body forces acting on the elemental parallelepiped. Then the resultant surface force in the positive x-direction is

$$-\frac{\partial P}{\partial x}\Delta x\Delta y\Delta z$$

By Newton's Second Law

$$-\frac{\partial P}{\partial x}\Delta x\Delta y\Delta z = \rho\ \Delta x\Delta y\Delta z\ a_x \tag{2-13}$$

where ρ = mean density in the parallelepiped.

a_x = the acceleration component in positive x-direction.

In general, the velocity components u,v, and w are functions of the coordinates and of the time t; thus

$$u = u(x,y,z,t)$$

where x,y,z, and t are independent variables.

Writing the total differential for u

$$du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz + \left(\frac{\partial u}{\partial t}\right)dt$$

Dividing by dt

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \tag{2-14}$$

Substituting Eq. (2-14) into (2-13)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{2-15}$$

where in the left hand side represents the acceleration of the medium in the x-direction and the right hand side the force per unit mass in the x-direction. There are no body forces acting. Here $\partial u/\partial t$, $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ are of the same order, but u,v,w are considered small so the product of $u(\partial u/\partial x)$, $v(\partial u/\partial y)$, and $w(\partial u/\partial z)$ can be neglected compared with $\partial u/\partial t$. The equation of motion in the x-direction then takes the simpler form.

$$\frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial P}{\partial x}$$

Substituting ψ into the above equation, multiply the equation by dx, then

$$\frac{\partial^2 \psi}{\partial x \partial t} dx = \frac{1}{\rho} \frac{\partial P}{\partial x} dx$$

After integrating

$$\frac{\partial \psi}{\partial t} = \int \frac{dP}{\rho}$$

Since ρ changes very little, it will be approximately correct to remove it from under the integral sign, calling it ρ_o, the mean density. Then $\int dP$ reduces simply to the excess pressure and is represented by p, the approximate relation is

$$\frac{\partial \psi}{\partial t} = \frac{p}{\rho_o} \qquad (2\text{-}16)$$

From the definition of sound velocity

$$c_o^2 = \frac{dP}{d\rho} = \frac{P - P_o}{\rho - \rho_o} = \frac{p}{\delta}$$

But

$$\delta = \rho_o s$$

by the definition of condensation, then there follows the important relationship

$$p = c_o^2 \rho_o s \qquad (2\text{-}17)$$

Eq. (2-16) now becomes

$$\frac{\partial \psi}{\partial t} = c_o^2 s \qquad (2\text{-}18)$$

Substituting Eq. (2-18) into (2-12) results

$$\frac{\partial^2 \psi}{\partial t^2} = c_o^2 \frac{\partial^2 \psi}{\partial x^2} \qquad (2\text{-}19)$$

which is the familiar equation of wave motion.

The equation just derived is perfectly general, its validity being limited only by the continuity of the medium and certain approximations requiring waves of small amplitude.

For a plane wave propagating in the x-direction, ψ is a function of x and t only

$$\frac{\partial^2\psi}{\partial t^2} = c_0^2 \frac{\partial^2\psi}{\partial x^2}$$

The general solution is

$$\psi = f(c_0t-x) + F(c_0t+x) \qquad (2\text{-}20)$$

wherein f and F are arbitrary functions. The first term on right-hand side represents a wave moving in the positive x direction. Thus the value of this term at $x = x_0$ for $t=0$, is $f(-x_0)$, and at $x=c_0t+x_0$ at the later time $t=t$, the value is

$$f\,\{c_0t-(c_0t+x_0)\} = f(-x_0)$$

That is, the disturbance at x_0 has traveled to x_0+c_0t in the time t. This means the disturbance progresses in the positive x-direction with velocity c_0. Similar reasoning will show that $F(x+c_0t)$ represents a wave propagating in the negative x-direction.

If the waves traveling in the positive direction were considered only and assuming that they were simple harmonic waves, then

$$\psi(x,t) = A\cos\frac{2\pi}{\lambda}(c_0t-x) = A\cos k_0(c_0t-x) \qquad (2\text{-}21)$$

where λ = wavelength or the distance between any two adjacent crests of a simple harmonic wave.

k_0 = wave number = $\frac{\omega}{c_0} = 2\frac{\pi}{\lambda}$.

ω = circular frequency

A = constant amplitude

From the definition of velocity potential

$$u = -\frac{\partial\psi}{\partial x} = -k_0 A \sin k_0(c_0 t - x) \qquad (2\text{-}22)$$

and from Eq. (2-18)

$$\frac{\partial\psi}{\partial t} = c_0^2 s = -Ak_0 c_0 \sin k_0(c_0 t - x)$$

$$s = -\frac{k_0}{c_0} A \sin(\omega t - k_0 x) \qquad (2\text{-}23)$$

The relation between acoustic pressure and condensation is

$$p = c_0^2 \rho_0 s$$

$$p = -k_0 c_0 \rho_0 A \sin(\omega t - k_0 x)$$

$$\frac{p}{u} = \rho_0 c_0 \qquad (2\text{-}24)$$

where $\rho_0 c_0$ is called the specific acoustic impedance.

D. Finite-Amplitude Wave Equation - Eulerian Description

I. Malecki (13) has derived a general wave equation by the method of small perturbations as follows:

Assume that

$$\psi = \psi_1 + \psi_2$$

where ψ_1 = the velocity potential satisfying the wave equation of the first approximation (small amplitude or linear terms).

ψ_2 = the corrective term describing the influence of the finite amplitude of the wave (second order terms).

Following the method of small perturbations as shown by Morse and Ingard (14), the following assumptions are made:

(1) The medium is an ideal gas at rest.

(2) Expansion and compression processes are isentropic.

(3) The excess pressure caused by the acoustic perturbation and particle velocity are not small compared with equilibrium static pressure and sound velocity.

(4) Plane waves are present.

For a plane wave propagating along the x-axis, the momentum equation is

$$\frac{\partial P}{\partial x} + \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = 0 \qquad (2\text{-}25)$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0 \qquad (2\text{-}26)$$

Eqs. (2-25) and (2-26) are expressed in Eulerian description. The corresponding equations in Lagrangian description are

$$\rho_o\left(\frac{\partial^2 \xi}{\partial t^2}\right) = -\frac{\partial P}{\partial x} \qquad (2\text{-}1)$$

$$\rho_o = \rho\left(1 + \frac{\partial \xi}{\partial x}\right) \qquad (2\text{-}2)$$

The thermodynamic relation (equation of state) for the adiabatic process is

$$\rho = \rho(P), \quad \frac{P}{P_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma}$$

The velocity of wave propagation is given by

$$c^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \qquad (2\text{-}28)$$

By the perturbation method

$$\psi = \psi_1 + \psi_2$$

$$P = P_o + p_1 + p_2$$

$$\rho = \rho_o + \rho_1 + \rho_2$$

$$u = u_1 + u_2$$

In these relations, the subscript zero represents the static or equilibrium state; the subscript one represents the first order or linear terms; and the subscript two represents second order or nonlinear terms.

To obtain the first order equation (the linear equation derived for infinitesimal amplitude waves), we consider only the first order terms in Eq. (2-26), (2-25) and (2-27)

$$\frac{\partial \rho_1}{\partial t} + \rho_o \frac{\partial u_1}{\partial x} = 0 \qquad (2\text{-}30)$$

$$\frac{\partial p_1}{\partial x} + \rho_o \frac{\partial u_1}{\partial t} = 0 \qquad (2\text{-}31)$$

$$\rho_1 = \left(\frac{\partial \rho}{\partial P}\right)_o p_1 = \frac{1}{c_o^2} p_1 \qquad (2\text{-}32)$$

Differentiating Eq. (2-30) with respect to t gives

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_o \frac{\partial^2 u_1}{\partial x \partial t} \qquad (2\text{-}33)$$

Differentiating Eq. (2-31) with respect to x

$$\frac{\partial^2 p_1}{\partial x^2} + \rho_o \frac{\partial^2 u_1}{\partial x \partial t} = 0 \qquad (2\text{-}34)$$

From Eq. (2-33) and (2-34)

$$\frac{\partial^2 p_1}{\partial x^2} = \frac{\partial^2 \rho_1}{\partial t^2} \qquad (2\text{-}35)$$

From Eq. (2-32) and (2-35)

$$\frac{\partial^2 p_1}{\partial x^2} = \frac{1}{c_o^2} \frac{\partial^2 p_1}{\partial t^2} \qquad (2\text{-}36)$$

The definition of velocity potential is

$$u_1 = - \frac{\partial \psi_1}{\partial x} \qquad (2\text{-}37)$$

Substituting Eq. (2-37) into (2-31)

$$p_1 = \rho_o(\frac{\partial\psi_1}{\partial t})$$

Eq. (2-36) can also be written as

$$\frac{\partial^2\psi_1}{\partial x^2} = \frac{1}{c_o^2}\frac{\partial^2\psi_1}{\partial t^2} \qquad (2\text{-}38)$$

To obtain the second order equation, consider only the second order terms (its magnitude is comparable to the square of first order terms) in Eqs.(2-25), (2-26) and (2-27)

$$\frac{\partial p_2}{\partial x} + \rho_o\frac{\partial u_2}{\partial t} + \rho_1\frac{\partial u_1}{\partial t} + \rho_o u\frac{\partial u_1}{\partial x} = 0 \qquad (2\text{-}39)$$

$$\frac{\partial \rho_2}{\partial t} + \rho_o\frac{\partial u_2}{\partial x} + \rho_1\frac{\partial u_1}{\partial x} + u_1\frac{\partial \rho_1}{\partial x} = 0 \qquad (2\text{-}40)$$

$$\rho_2 = (\frac{\partial\rho}{\partial P})_o\ p_2 + \frac{1}{2}(\frac{\partial^2\rho}{\partial P^2})_o p_1^2 = \frac{1}{c_o^2}\ p_2 + \frac{(1-\gamma)}{2\rho_o c_o^4}\ p_1^2 \qquad (2\text{-}41)$$

Differentiating Eq. (2-39) with respect to x

$$\frac{\partial^2 p_2}{\partial x^2} + \rho_o\frac{\partial^2 u_2}{\partial x\partial t} + \frac{\partial\rho_1}{\partial x}\frac{\partial u_1}{\partial t} + \rho_1\frac{\partial^2 u_1}{\partial x\partial t}\ \rho_o(\frac{\partial u_1}{\partial x})^2 + \rho_o u_1\frac{\partial^2 u_1}{\partial x^2} = 0 \qquad (2\text{-}42)$$

Differentiating Eq. (2-40) with respect to t

$$\frac{\partial^2 \rho_2}{\partial t^2} + \rho_o\frac{\partial^2 u_1}{\partial x\partial t} + \frac{\partial\rho_1}{\partial t}\frac{\partial u_1}{\partial x} + \rho_1\frac{\partial^2 u_1}{\partial x\partial t} + u_1\frac{\partial^2\rho_1}{\partial x\partial t} + \frac{\partial u_1}{\partial t}\ \frac{\partial\rho_1}{\partial x} = 0 \qquad (2\text{-}43)$$

Subtracting Eq. (2-43) from Eq. (2-42), and cancelling the identical terms

$$\frac{\partial^2 p_2}{\partial x^2} - \frac{\partial^2\rho_2}{\partial t^2} + \{\rho_o(\frac{\partial u_1}{\partial x})^2 + \rho_o u_1\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial\rho_1}{\partial t}\frac{\partial u_1}{\partial x} - u_1\frac{\partial^2\rho_1}{\partial x\partial t}\} = 0 \qquad (2\text{-}44a)$$

Rearranging (2-44a) gives

$$\frac{\partial^2 p_2}{\partial x_2} - \frac{\partial^2\rho_2}{\partial t_2} + \{\rho_o\frac{\partial}{\partial x}(u_1\frac{\partial u_1}{\partial x}) - \frac{\partial}{\partial x}(u_1\frac{\partial\rho_1}{\partial t})\} = 0 \qquad (2\text{-}44)$$

From Eq. (2-30)

$$\frac{\partial \rho_1}{\partial t} = -\rho_o \frac{\partial u_1}{\partial x}$$

Substituting into Eq. (2-44) gives

$$\frac{\partial^2 \rho_2}{\partial x^2} - \frac{\partial^2 \rho_2}{\partial t^2} - 2\frac{\partial}{\partial x}(u_1 \frac{\partial \rho_1}{\partial t}) = 0 \qquad (2\text{-}45)$$

Substituting the relation in Eq. (2-41) into Eq. (2-45)

$$\frac{\partial^2 p_2}{\partial x^2} - \frac{1}{c_o^2}\frac{\partial^2 p_2}{\partial t^2} - \frac{(1-\gamma)}{2\rho_o c_o^4}\frac{\partial^2}{\partial t^2}(p_1)^2 - 2\frac{\partial}{\partial x}(u_1 \frac{\partial \rho_1}{\partial t}) = 0 \qquad (2\text{-}46)$$

Substituting the following relations into Eq. (2-46)

$$p_1 = \rho_o \frac{\partial \psi_1}{\partial t}, \quad p_2 = \rho_o \frac{\partial \psi_2}{\partial t}, \quad u_1 = -\frac{\partial \psi_1}{\partial x}, \quad \rho_1 = \frac{\rho_o}{c_o^2}\frac{\partial \psi_1}{\partial t}$$

which gives

$$\frac{\partial^2}{\partial x^2}(\rho_o \frac{\partial \psi^2}{\partial t}) - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}(\rho_o \frac{\partial \psi_2}{\partial t}) - \frac{(1-\gamma)}{2\rho_o {}_o^4}\frac{\partial^2}{\partial t^2}(\rho_o \frac{\partial \psi_1}{\partial t})^2 - 2\frac{\partial}{\partial x}\{(-\frac{\partial \psi_1}{\partial x})\frac{\partial}{\partial x}(\frac{\rho_o}{c_o^2}$$

$$\frac{\partial \psi_1}{\partial t})\} = 0 \qquad (2\text{-}47)$$

It is desired to arrange the above equation into a form as $\frac{\partial}{\partial t}\{\ \}$.

Starting with the first term, we can write:

$$\frac{\partial^2}{\partial x^2}(\rho_o \frac{\partial \psi_2}{\partial t}) = \frac{\partial}{\partial t}(\rho_o \frac{\partial^2 \psi_2}{\partial x^2})$$

Assuming that ψ_1, ψ_2 have continuous derivative, this justifies the interchanging of the order of differentiation.

The second term

$$-\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}(\rho_o \frac{\partial \psi_2}{\partial t}) = \frac{\partial}{\partial t}(-\frac{\rho_o}{c_o^2}\frac{\partial^2 \psi_2}{\partial t^2})$$

The third term

$$-\frac{(1-\gamma)}{2\rho_o c_o^2}\frac{\partial^2}{\partial t^2}(\rho_o \frac{\partial \psi_1}{\partial t})^2 = \frac{\partial}{\partial t}\{\frac{(\gamma-1)}{2c_o^4}\rho_o \frac{\partial}{\partial t}(\frac{\partial \psi_1}{\partial t})^2\} = \frac{\partial}{\partial t}\{\frac{(\gamma-1)}{2\ c_o^4}\rho_o 2\frac{\partial \psi_1}{\partial t}\frac{\partial^2 \psi_1}{\partial t^2}$$

$$= \frac{\partial}{\partial t}\{\frac{(\gamma-1)\rho_o}{c_o^4}\frac{\partial \psi_1}{\partial t}(c_o^2 \frac{\partial^2 \psi_1}{\partial x^2})\} = \frac{\partial}{\partial t}\{\frac{(\gamma-1)}{c_o^2}\frac{\partial \psi_1}{\partial t}\frac{\partial^2 \psi_1}{\partial x^2}\}$$

The fourth term

$$-2\frac{\partial}{\partial x}\{(-\frac{\partial\psi_1}{\partial x})\ \frac{\partial}{\partial t}(\frac{\rho_o}{c_o^2}\ \frac{\partial\psi_1}{\partial t})\} = \frac{2\rho_o}{c_o^2}\ \frac{\partial}{\partial x}\ \{\frac{\partial\psi_1}{\partial x}\ \frac{\partial^2\psi_1}{\partial t^2}\} = 2\ \frac{\rho_o}{c_o^2}\ \frac{\partial}{\partial x}\{\psi_{1x}\psi_{1tt}\}$$

$$=\frac{2\rho_o}{c_o^2}\{\psi_{1xx}\psi_{1tt}+\psi_{1x}\ \psi_{1ttx}\}= \frac{2\rho_o}{c_o^2}\ \{\psi^2_{1xt}+\psi_{1x}\ \psi_{1xtt}\}$$

$$= \frac{\partial}{\partial t}\{\frac{2\rho_o}{c_o^2}(\psi_{1x}\ \psi_{1xt})\}$$

where the identity

$$\psi_{1xx}\ \psi_{1tt} = (\psi_{1xt})^2 \tag{2-48}$$

is derived as follows.

In Eq. (2-24), a relation between particle velocity and sound pressure was shown as

$$u = \frac{p}{\rho_o c_o} \tag{2-24}$$

which is true for linear terms of p and u. Into this equation, the following relations are substituted:

$$u_1 = -\frac{\partial\psi_1}{\partial x}$$

$$p_1 = \rho_o\frac{\partial\psi_1}{\partial t}$$

The result is

$$-\frac{\partial\psi_1}{\partial x} = \frac{\rho_o\frac{\partial\psi_1}{\partial t}}{\rho_o c_o} = \frac{1}{c_o}\ \frac{\partial\psi_1}{\partial t}$$

Thus

$$\frac{\partial^2\psi_1}{\partial t^2} = = c_o\frac{\partial^2\psi_1}{\partial x\partial t} \tag{2-48a}$$

Comparing with Eq. (2-38)

$$\frac{\partial^2\psi_1}{\partial x^2} = \frac{1}{c_o^2}\ \frac{\partial^2\psi_1}{\partial t^2} \tag{2-38}$$

Multiplying Eq. (2-38) with $\frac{\partial^2\psi_1}{\partial t^2}$, and taking consideration of Eq. (2-48a), it becomes

$$\frac{\partial^2\psi_1}{\partial x^2}\ \frac{\partial^2\psi_1}{\partial t^2} = \frac{1}{c_o^2}\ (\frac{\partial^2\psi_1}{\partial t^2})^2 = \frac{1}{c_o^2}\ (-c_o\frac{\partial^2\psi_1}{\partial x\partial t})^2 = (\frac{\partial^2\psi_1}{\partial x\partial t})^2 \tag{2-48}$$

This proves the identity of Eq. (2-48)

The final result of Eq. (2-47) is

$$\frac{\partial^2 \psi_2}{\partial t^2} = c_o^2 \frac{\partial^2 \psi_2}{\partial x^2} + (\gamma - 1) \frac{\partial \psi_1}{\partial t} \frac{\partial^2 \psi_1}{\partial x^2} + 2 \frac{\partial \psi_1}{\partial x} \frac{\partial^2 \psi_1}{\partial x \partial t} \quad (2\text{-}49)$$

III. THE APPROXIMATE SOLUTION OF FINITE AMPLITUDE WAVE EQUATION

A. Approximate Solutions

The Eq. (2-50b) is a non-linear partial differential equation. In order to find an analytical solution, the procedure that Lamb has used to obtain an approximate solution for finite amplitude waves will be introduced as follows.

An approximate solution of the finite amplitude wave equation in the Lagrangian description has been obtained by Lamb (15). He started from Eq. (2-6) as shown

$$\frac{\partial^2 \xi}{\partial t^2} = c_o^2 \ (1+\frac{\partial \xi}{\partial x})^{-(\gamma+1)} \ \frac{\partial^2 \xi}{\partial x^2} \qquad (2\text{-}6)$$

From a binomial series expansion of the term $(1+\frac{\partial \xi}{\partial x})^{-(\gamma+1)}$ neglecting higher order terms

$$(1+\frac{\partial \xi}{\partial x})^{-(\gamma+1)} \cong 1-(\gamma+ 1) \ \frac{\partial \xi}{\partial x} \qquad (3\text{-}1)$$

then Eq. (2-6) becomes

$$\frac{\partial^2 \xi}{\partial t^2} - c_o^2 \ \frac{\partial^2 \xi}{\partial x^2} = -c_o^2 (\gamma+1) \ \frac{\partial \xi}{\partial x} \ \frac{\partial^2 \xi}{\partial x^2} \qquad (3\text{-}2)$$

this approximation restricts the dilatation $\frac{\partial \xi}{\partial x}$ to values small compared with γP_o which is a direct result from Eq. (3-4), to be derived below.

With the boundary condition $\xi = A\cos\omega t$ imposed at $x=0$, and assuming a complete absence of reflections, the first order solution of Eq. (3-2) is obtained by neglecting the second order correction term on the right hand side of Eq. (3-2). It is

$$\xi = A\cos\omega(t-\frac{x}{c})$$

Substituting the first order solution into the right hand side of Eq. (3-2), and solve the particular solution of Eq. (3-2), the total solution will be the approximate solution of Eq. (3-2) as

$$\xi = A\cos\omega(t-\frac{x}{c_o}) + \frac{\gamma+1}{8}\frac{\omega^2}{c_o^2}A^2x\{1-\cos 2\omega(t-\frac{x}{c_o})\} \quad (3\text{-}3)$$

Thuras, Jenkins and O'Neil (16) continued the work to the solution of sound pressure. Combining Eq. (2-1) and Eq. (2-5) gives

$$P = P_o(\frac{\rho}{\rho_o})^{\gamma} = P_o(1+\frac{\partial\xi}{\partial x})^{-\gamma}$$

After expansion of $(1+\frac{\partial\xi}{\partial x})^{-\gamma}$ into a binomial series

$$P \cong P_o(1-\frac{\partial\xi}{\partial x}) \quad (3\text{-}4)$$

Neglecting the terms of small amplitude in the region where $4\pi x$ is large compared with the wavelength λ, Eq. (3-4) may be written as

$$P = P_{dc} + p_1 + p_2$$

where

$$P_{dc} = P_o - \frac{\gamma P_o}{8}(\gamma+1)k_o^2A^2 * \quad (3\text{-}5)$$

$$p_1 = -\gamma P_o k_o A\sin(\omega t - k_o x) \quad (3\text{-}6)$$

$$p_2 = \frac{(\gamma+1)}{4}A^2\gamma P_o k_o^3 x \sin 2(\omega t - k_o x) \quad (3\text{-}7)$$

p_1 and p_2 are the fundamental and second harmonic of acoustic pressure.

I. Malecki (17) solved Eq. (3-2) in a slightly different way as follows

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*P_{dc} is defined and discussed in Eq. (3-39)

Corresponding to the piston displacement (source motion) as

$$\xi = A(1-\cos\omega t) \tag{3-8}$$

the first and second order solution of Eq. (3-2) is obtained by using Lamb's approximation method.

$$\xi_1 = A\{1-\cos(\omega t-k_o x)\} \tag{3-9}$$

$$\xi_2 = \frac{A^2}{8}(\gamma+1)k_o^2 x\{1-\cos 2(\omega t-k_o x)\} \tag{3-10}$$

where $k_o=\frac{\omega}{c_o}$ is the wave number for a wave with an infinitesimal amplitude.

After differentiation of Eq. (3-9), and Eq. (3-10) with respect to time, the acoustic velocities are

$$u_1 = \omega A\sin(\omega t-k_o x) \tag{3-11}$$

$$u_2 = \frac{1}{4}\omega A^2 x(\gamma+1)k_o^2 \sin 2(\omega t-k_o x) \tag{3-12}$$

Malecki (22) derived an equation for finite amplitude wave propagation as shown

$$P = P_o\left(1+\frac{\gamma+1}{2}\frac{u}{c_o}\right)^{\frac{2\gamma}{(\gamma-1)}} \tag{3-13}$$

Expanding Eq. (3-13) into a binomial series, with the assumption that $u_{max}<<c_o$, the approximate form is

$$P \cong P_o\left(1+\gamma\frac{u}{c_o} + \frac{\gamma(\gamma+1)}{4}\frac{u^2}{c_o^2}\right) \tag{3-14}$$

Substituting Eq. (3-11) and Eq. (3-12) into Eq. (3-14), the first and second order acoustic pressures are

$$p_1 = \gamma P_o k_o A\sin(\omega t-k_o x) \tag{3-15}$$

$$p_2 = 2p_{re}\left\{\frac{1}{2}\{1-\cos 2(\omega t-k_o x)\} + k_o x \sin 2(\omega t-k_o x)\right\} \tag{3-16}$$

where

$$p_{re} = \frac{1}{8} P_0 k_0^2 A^2 \gamma \ (\gamma+1) = \frac{1}{8} \rho_0 \omega^2 A^2 \ (\gamma+1)$$

Eqs. (3-15) and (3-16) are the approximate solution of the finite-amplitude wave equation, Eq. (3-2)

The finite-amplitude wave equation in the Eulerian description, given in Eqs. (2-38) and (2-49), is repeated below

$$\frac{\partial^2\psi_1}{\partial t^2} - c_0^2 \frac{\partial^2\psi_1}{\partial x^2} = 0, \text{ for the linear term, and} \tag{3-18}$$

$$\frac{\partial^2\psi_2}{\partial t^2} - c_0^2 \frac{\partial^2\psi_2}{\partial x^2} = (\gamma-1)\frac{\partial\psi_1}{\partial t}\frac{\partial^2\psi_1}{\partial x^2} + 2\frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_1}{\partial x\partial t}, \text{ for the first order nonlinear correction.} \tag{3-19}$$

Assuming no reflections and a sinusoidal particle velocity $u_1 = b\sin\omega t$ at the boundary $x = 0$, the solution of Eq. (3-18) is

$$\psi_1(x,t) = -\frac{b}{k_0}\cos(\omega t - k_0 x) \tag{3-20}$$

where b is a constant

Substituting Eq. (3-20) into the right hand side of Eq. (3-19) gives

$$\frac{\partial^2\psi_2}{\partial t^2} - c_0^2 \frac{\partial^2\psi_1}{\partial x^2} = \frac{(\gamma+1)}{2} \ b^2\omega\sin 2(\omega t - k_0 x) \tag{3-21}$$

Assuming the boundary condition is $\psi_2 = 0$, the particular solution of ψ_2 in Eq. (3-21) is

$$\psi_2 = -Kx - \frac{(\gamma+1)b^2 x}{8} \cos 2(\omega t - k_0 x) \tag{3-22}$$

where K is an undetermined constant.

Then, the particle velocity is $u_1 + u_2$,

where

$$u_1 = -\frac{\partial\psi_1}{\partial x} = b\sin(\omega t - k_0 x) \tag{3-23}$$

$$u_2 = -\frac{\partial\psi_2}{\partial x} = K + \frac{(\gamma+1)b^2}{8c_0}\{\cos 2(\omega t-k_0x)-2k_0x\ \sin 2(\omega t-k_0x)\}$$

By changing the constant displacement amplitude in Eqs. (3-15) and (3-16) into a corresponding pressure amplitude D, the acoustic pressure in the Lagrangian description can be written as

$$p_1 = D\sin(\omega t-k_0x) \tag{3-24}*$$

$$p_2 = p_{r_L} + p_2' = \frac{(\gamma+1)D^2}{8\rho_0c_0^2} - \frac{(\gamma+1)D^2}{8\rho_0c_0^2}\{\cos 2(\omega t-k_0x)-2k_0x\sin 2(\omega t-k_0x)\}$$

By changing the constant velocity amplitude in Eq. (3-23), the corresponding acoustic pressure in the Eulerian description can be written as

$$p_1 = D\sin(\omega t-k_0x)$$

$$p_2 = p_{re} + p_2' = K + \frac{(\gamma+1)D^2}{8\rho_0c_0^2}\{\cos 2(\omega t-k_0x)-2k_0x\ \sin 2(\omega t-k_0x)\} \tag{3-25}*$$

By comparison of Eq. (3-24) and Eq. (3-25), K is the undetermined constant pressure. The second harmonic sound pressures are the same in Eq. (3-24) and Eq. (3-25); this justifies the Lagrangian formulation used to compare experimental results with theoretical predictions in the work reported by Thuras, Jenkins, and O'Neil (16).

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* p_2' is the second harmonic sound pressure.

p_{re} and p_{r_L} are defined in Section C of this chapter

B. Nonlinear Distortion Coefficient.

One of the methods to investigate the effect of nonlinearity on the acoustic wave propagation is to measure the nonlinear distortion coefficient. The nonlinear distortion coefficient h is defined as the ratio of the amplitudes of the second harmonic sound pressure p_2'. The acoustic pressure input is a sinusoidal wave. If the fundamental and second harmonic only were considered, then when h = 50%, the sinusoidal wave deforms into an approximation of a wave.

From Eq. (3-24) and Eq. (3-25), the nonlinear distortion coefficient derived from the Eulerian and Lagrangian formulations is the same, that is

$$h_L = h_e = \frac{p'_{2max}}{p_{1max}} = \frac{D(1+4k_0^2x^2)^{\frac{1}{2}}}{4\rho_0 c_0^2} \qquad (3\text{-}26)$$

C. Radiation Pressure

From Eq. (3-25), the time average of p_1 and p_2 is a constant value of pressure called the Rayleigh radiation pressure p_{re},

where

$$p_{re} = K \qquad (3\text{-}27)$$

This pressure is defined to be the difference between the time averages of the pressure at any point of a fluid traversed by a compressional wave and the pressure which would have existed in a fluid of the same mean density at rest.

From Eq. (3-24), the time average of p_1 and p_2 can be written as

$$p_{r_L} = \frac{(\gamma+1)D^2}{8\rho_o c_o^2} \tag{3-28}$$

The P_{dc} term in Eq. (3-5) can be written as

$$P_{dc} = P_o + p_{r_L} \tag{3-29}$$

The radiation pressure that is more practical from the experimental point of view is generally known as the "Langevin" radiation pressure and is defined as the difference between the pressure at a wall and the pressure in the medium at rest behind the wall. If an ideally absorbing surface is suspended so as to intercept an acoustic beam, the requirement of continuity of displacement means that the particles of the surface must follow the motion of the particle of the fluid. In this arrangement, averaging of the pressure must be taken not at a fixed point (Eulerian coordinates) but over the cycle undergone by a particular particle originally at rest (Lagrangian coordinates). From this argument, it follows that Eq. (3-28) gives the Langevin radiation pressure.

The radiation pressure is a constant term. In this thesis, only the acoustic effects of alternating (audible) pressures are considered. Furthermore, for those frequencies and distances where p'_{2max} is not negligible as compared to p_{1max}, p_{r_L} is at least an order of magnitude less than p'_{2max}.

D. Numerical Solution by Method of Characteristics

The numerical solution is another alternative method to investigate the effect of finite-amplitude wave propagation on the change of waveform. It is possible to set up a general computer program to evaluate the effects of finite amplitude wave and steady flow on the plane sound wave propagation.

Starting from the hydrodynamic equations in the Eulerian description,

$$\frac{\partial P}{\partial x} + \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = 0 \tag{3-30}$$

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0 \tag{3-31}$$

Assuming the medium is ideal gas and that the acoustic propagation is adiabatic

$$\frac{\rho}{\rho_o} = \left(\frac{P}{P_o}\right)^{\frac{1}{\gamma}} \tag{3-32}$$

By using the relation in P and ρ in Eq. (3-32) to change the partial derivatives $\frac{\partial \rho}{\partial t}$ and $\frac{\partial \rho}{\partial x}$ to $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial x}$, Eq. (3-30) and Eq. (3-31) can be transformed into a system of equations with two dependent variables P and u. The standard method of characteristics for hyperbolic partial differential equations yields the two characteristics α and β.

$$\alpha \begin{cases} \dfrac{dx}{dt} = u+c_o \left(\dfrac{P}{P_o}\right) \dfrac{(\gamma-1)}{2\gamma} \\ \\ \gamma P_{du} + c_o \left(\dfrac{P}{P_o}\right) \dfrac{(\gamma-1)}{2\gamma} dP = 0 \end{cases} \tag{3-33}$$

$$
\beta \begin{cases} \dfrac{dx}{dt} = u - c_o \left(\dfrac{P}{P_o}\right)^{\frac{(\gamma-1)}{2\gamma}} \\[2ex] P_{du} - c_o \left(\dfrac{P}{P_o}\right)^{\frac{(\gamma-1)}{2\gamma}} dP = 0 \end{cases} \tag{3-34}
$$

From Eq. (3-33) and Eq. (3-34), applying the procedures of Lister's paper (18), the total pressure and particle velocity as functions of x and t can be calculated. The factors of sound amplitude and steady flow velocity may be supplied as input boundary conditions, assuming that the acoustic velocity caused by the applied sound pressure may be superposed on the steady flow velocity.

IV. TRIMMER'S INVESTIGATION ON THE EFFECT OF STEADY FLOW.

John D. Trimmer (19) theoretically investigated the problem of the behavior of small-amplitude sound waves in a pipe through which the medium is moving in a unidirectional flow with uniform velocity V.

He started from the velocity potential

$$\psi_x = u + V \tag{4-1}$$

where $u = u(x,t)$, small in amplitude, and V is a constant, not necessarily small.

The equation of motion, written in terms of ψ, is

$$\psi_{xt} + \psi_x \psi_{xx} = (-\frac{1}{\rho})\ p_x \tag{4-2}$$

Similiarly, the equation of continuity is

$$\rho_t + (\rho\psi_x)_x = 0 \tag{4-3}$$

By putting $\rho = \rho_o(1+s)$ and neglecting product $s\psi_{xx}$, Eq. (4-3) may be written as

$$s_t + \psi_{xx} + s_x \psi_x = 0 \tag{4-4}$$

Multiplying through Eq. (4-2) by dx, integrating and putting $(\frac{P}{\rho_o}) = \int (\frac{1}{\rho})\ p_x dx$, one gets

$$\psi_t + \frac{1}{2}\psi_x^2 = -\ (\frac{P}{\rho_o}) \tag{4-5}$$

With the help of the relation $p = \rho_o\ c^2 s$, Eqs. (4-4) and (4-5) combined into

$$\psi_{tt} = (c_o^2 - \psi_x^2)\psi_{xx} - 2\psi_x \psi_{xt} \tag{4-6}$$

Two limiting cases may now be discussed. Referring to the expression (4-1) for the velocity, if V is about the same magnitude as u, then Eq. (4-6) may as well be replaced by the simple wave equation $\psi_{tt} = c_o^2\psi_{xx}$. On the other hand, if V is quite large compared to u, one may replace ψ_x by V in Eq. (4-1), and to this approximation Eq. (4-6) changes to the form

$$\psi_{tt} = (c_o^2 - V^2)\psi_{xx} - 2V\psi_{xt} \tag{4-7}$$

which is equivalent to transforming the simple wave equation

$$\psi_{t't'} = c_o^2\ \psi_{x'x'} \tag{4-8}$$

by

$$x = x'+Vt',\ t = t' \tag{4-9}$$

with the solution

$$\psi = f(x-c_1t) + g(x+c_2t)+Vx \tag{4-10}$$

where $c_1 = c+V,\ c_2 = c-V$

When acoustic effects are being considered, the Eq. (4-10) may be written as

$$\psi(x,t) = f(x-c_1t) + g(x+c_2t) \tag{4-11}$$

or

$$\psi(x,t) = Ae^{i(\omega t-k_1x)}+Be^{i(\omega t+k_2x)} \tag{4-12}$$

Where A and B are constants.

Compared with the solution for Eq. (4-8)

$$\psi(x,t) = Ae^{i(\omega t-kx)}+Be^{i(\omega t+kx)} \tag{4-13}$$

the effect of the steady flow will be represented by the modification of wave number k. Ronneberger (20) used the same equation, (4-12), in his investigation of sound in air flowing through tubes.

V. DISCUSSION AND CONCLUSION

The purpose of this research was to analyze the effects of intense sound levels and steady flow on the propagation of plane waves in air. The analysis has been limited to the study of the change of wave profile due to weak nonlinearities caused by intense sound levels and steady flow.

First, the differences and advantages of the Lagrangian and Eulerian descriptions were considered and evaluated. It is shown that for comparison with experimental measurements, the Eulerian description is preferred.

Formulation of the finite-amplitude wave equation both in Lagrangian and Eulerian coordinates has been introduced. The finite-amplitude wave equation in Lagrangian coordinates was derived as a classic example. The finite-amplitude wave equation in Eulerian coordinates was derived by the perturbation method where only the first and second-order acoustic variables were considered. The approximate solution consists of the simple wave solution and a correction term represented by the second harmonic.

Trimmer (19) concluded that the effect of steady flow on infinitesimal amplitude sound propagation could be obtained by modifying the sound velocity in the elementary wave propagation solution. The same idea has been extended to finite-amplitude wave propagation by modifying the sound velocity such that the combined effects of finite- amplitude (for weak nonlinearities) and steady flow can be evaluated.

If a high-intensity sound wave is produced by a source, the second harmonic sound pressure generated along the path of propagation may not necessarily be ignored. Consider, for example, a sound source generating a plane sinusoidal sound pressure at 120 dB and 100 Hz into a pipe 1 meter in length. The distortion coefficient (sound pressure ratio of second harmonic to fundamental as defined in Eq. (3-26)), in this case is -73 dB, which is negligible. However, when the source sound pressure is 160 dB and 1000 Hz and the pipe length is 5 meters, the distortion coefficient is 0.75 dB, which is significant. The magnitude of the generated second harmonic sound pressure due to nonlinearity of the medium is proportional to source frequency, distance from the source and the square of the source sound pressure. The magnitudes of the second harmonic sound pressure vs. distance from the source at different source levels and frequencies are shown in Figure 4.

When steady flow exists, first order correction factors for the finite amplitude distortion coefficients are obtained by modifying the sound velocity. Figure 5-1 shows the correction factor when the steady flow is in direction of sound propagation, and Figure 5-2 shows the correction factor when the steady flow is opposite to the direction of sound propagation. These correction factors may either be positive or negative, depending on the steady flow direction. They do not include effects of noise generated by turbulence or flow separation.

From Eq. (3-26) it is shown that the fundamental sound pressure amplitude remains the same while the second harmonic sound pressure increases proportional to the distance from source. This is an apparent violation of conservation of energy which results from the approximate nature of the solution. However, the violation is relatively minor in most cases. For example, in a 5 meter tube, the magnitude of the second harmonic sound pressure is less than 10% of the fundamental sound pressure for a 140 dB, 500 Hz source (Figure 4). Since the acoustic energy is proportional to the square of the r.m.s. acoustic pressure, the acoustic energy of the second harmonic is only 1% of the energy related to the fundamental sound pressure.

The approximate solution of the nonlinear wave equation is limited to weak nonlinearities; this means that as the sound levels exceed 140 dB the prediction of the magnitude of second harmonic sound pressure may be lower than the true value. For this reason, the data of Figure 4 apply with greatest accuracy to regions where weak-nonlinearities exist; i.e. where h the ratio p'_{2max}/p_{1max} is about 10 percent or less. However, effects of viscosity and heat conduction at the tube walls (which have been neglected to this point) tend to increase the range of application by reducing the relative magnitude of the second harmonic. As suggested in A. Fay's work (21), the attenuation due to heat conduction and viscosity is proportional to square root of frequency, which means that the second harmonic will be attenuated more than the fundamental.

Theoretical evaluation of the effect of attenuation due to viscosity and heat conduction losses at tube walls is shown in Figure 6. The first order correction factor of the nonlinear distortion coefficient is a function of source frequency, distance from the sound source, and tube diameter but independent of source amplitude. The calculation procedure is the same as described by Thuras, Jenkins and O'Neil (16), in which theoretical attenuation factors are applied separately to the fundamental and second harmonic components. For example, in a 2 in. diameter tube 5 meters long, the reduction in the distortion coefficient from tube losses is 1.3 dB(Figure 6-2), when source frequency is 1000 Hz.

Experimental measurements of the distortion coefficients can be obtained by using a long tube with a suitable anechoic termination. Sound pressures within the tube at various locations would be measured by probe microphones. An acoustic driver at the tube inlet would generate a sinusoidal sound wave, and an electronic filter would be used to measure the sound pressure at the frequency of interest. In the case that steady flow exists, the flow noise should be kept at a minimum such that the measurement of low sound levels will not be masked.

This thesis is part of a continuing program to improve the design of acoustic silencers. An important aspect of this program is to establish the effects of steady flow and intense sound levels on results predicted from elementary acoustic theory. The research described in this thesis indicates that flow and amplitude effects for plane sound waves propagating in tubes of constant diameter become increasingly significant with increasing sound level, frequency and tube length. Attenuation due to tube wall losses tends to lessen the significance of nonlinear effects, while steady flow may enhance or diminish these effects depending on its direction.

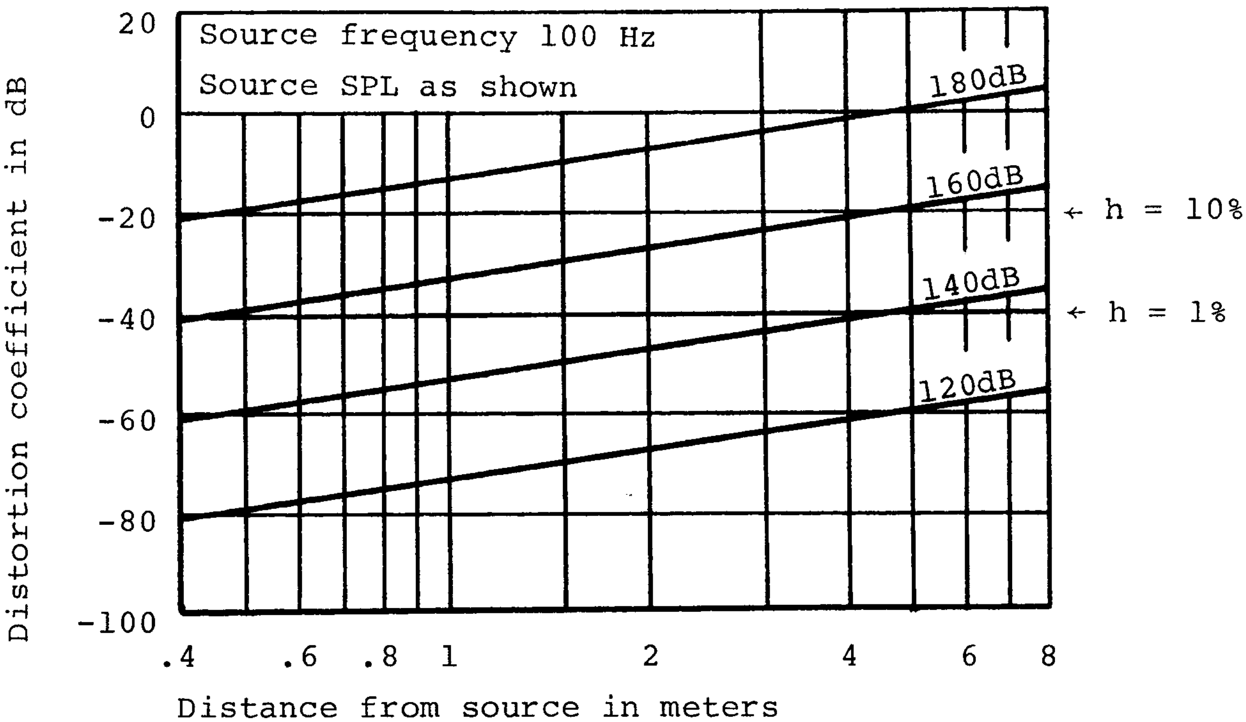

Fig. 4-1. Finite-amplitude distortion coefficient for 100 Hz source.

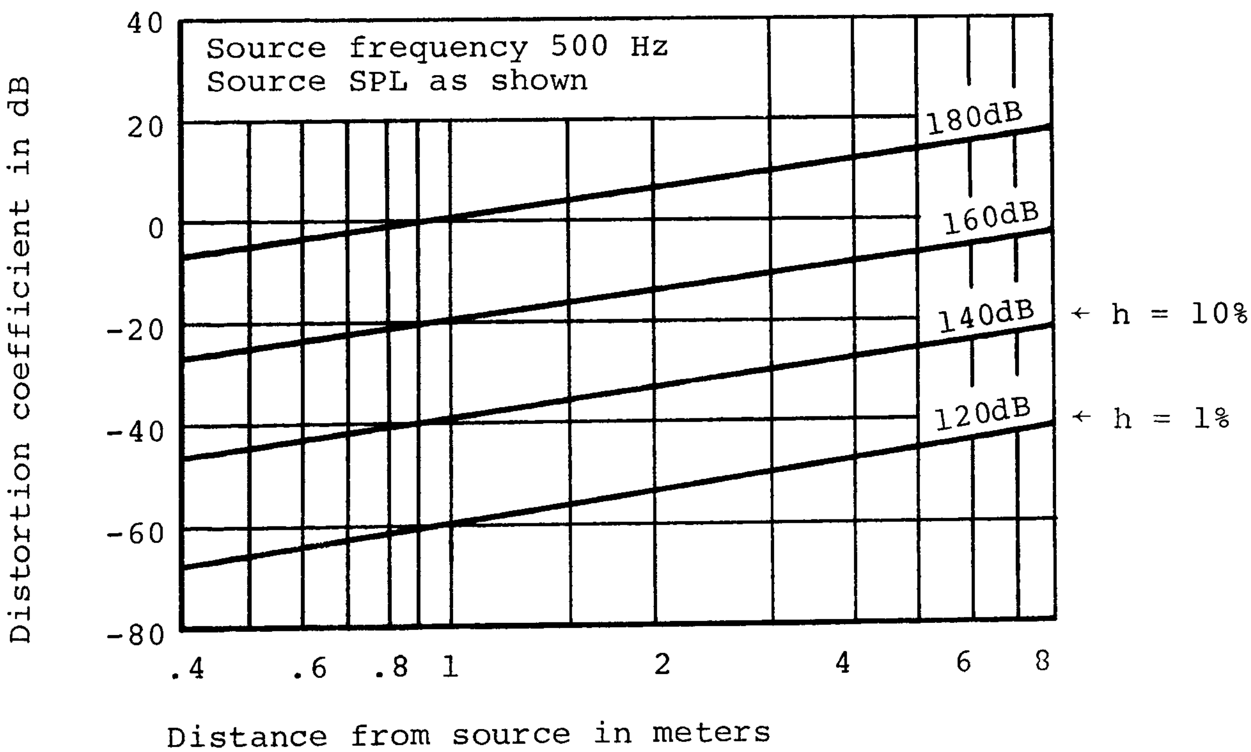

Fig. 4-2. Finite-amplitude distortion coefficient for 500 Hz source.

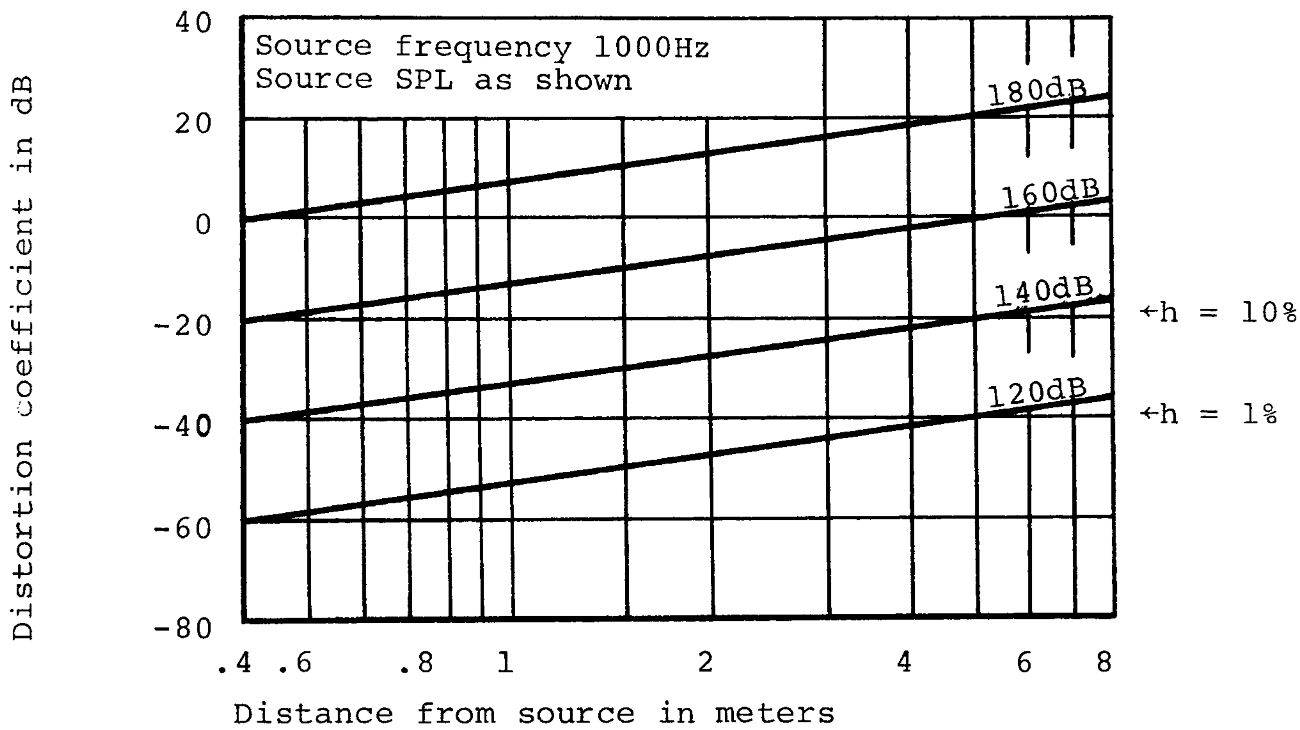

Fig. 4-3. Finite-amplitude distortion coefficient for 1000Hz source.

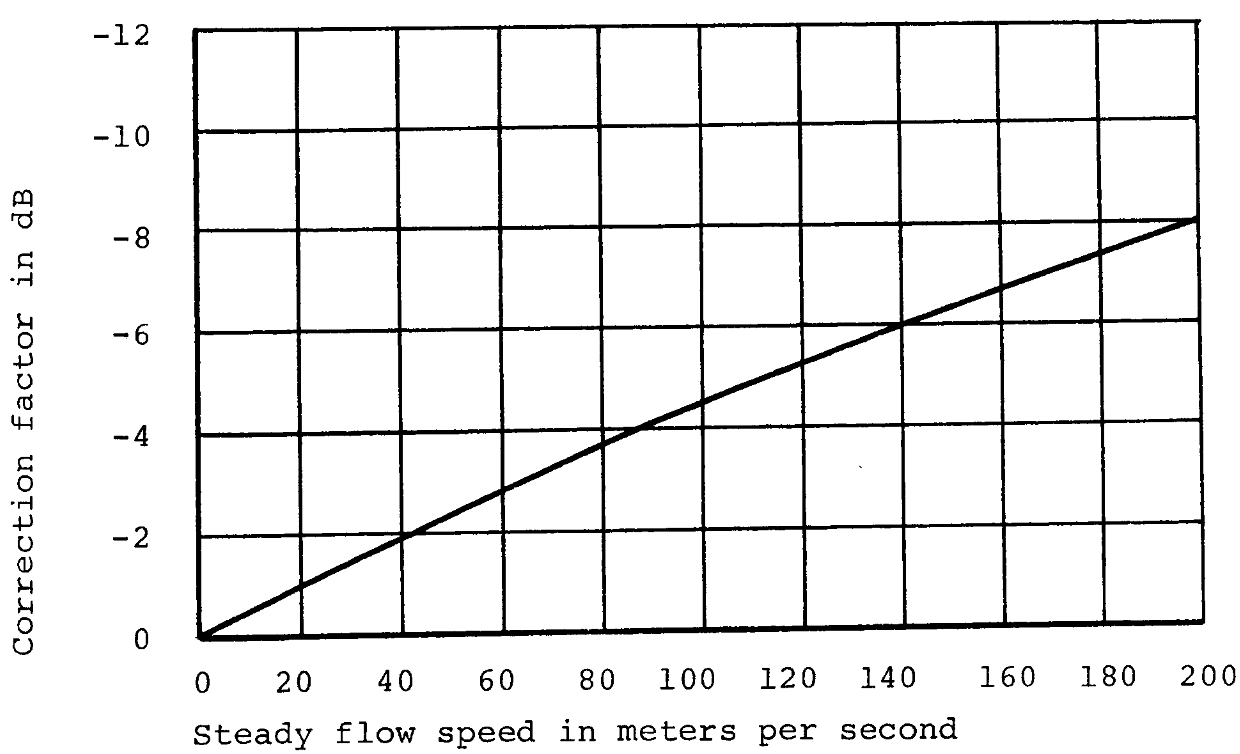

Fig. 5-1. Correction factor for data of Fig. 4, for steady flow in direction of sound propagation.

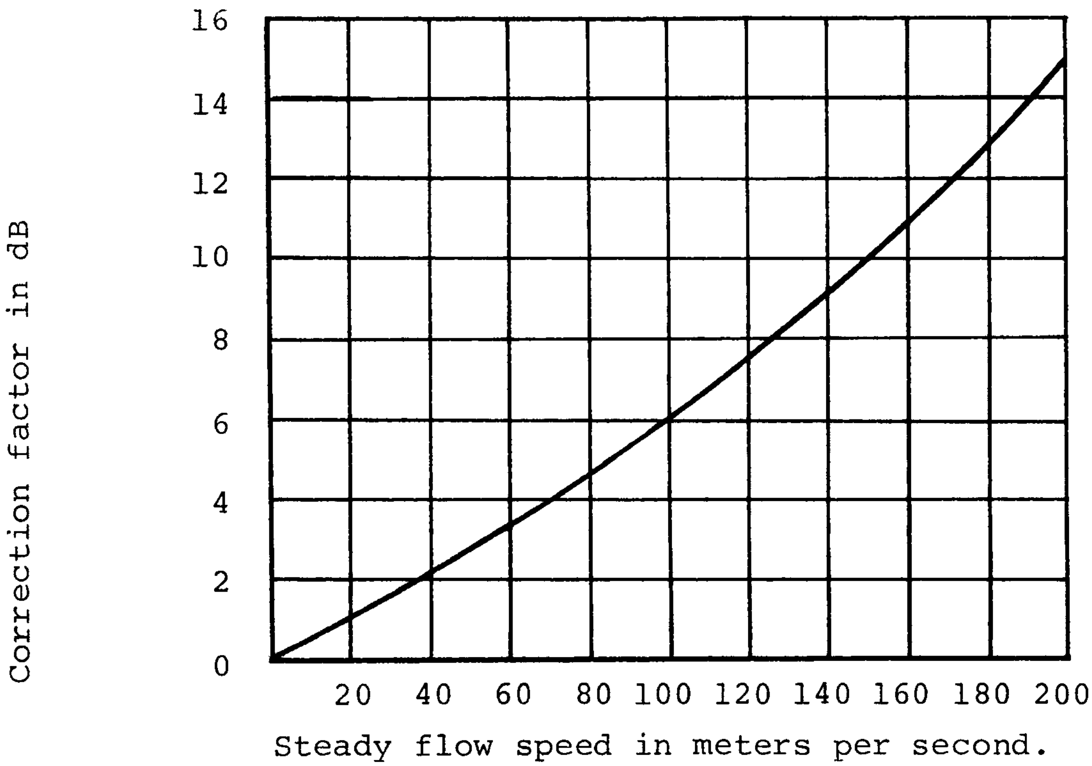

Fig. 5-2. Correction factor for data of Fig. 4, for steady flow opposite to direction of sound propagation.

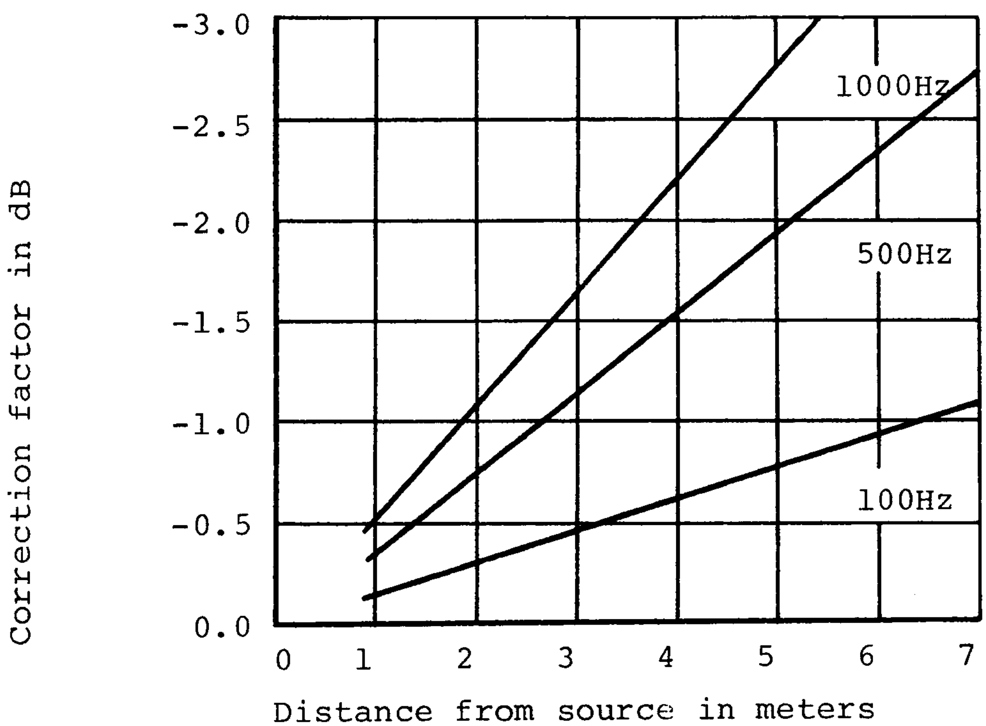

Fig. 6-1. Correction factor for data of Fig.4, for attenuation in 1 in. dia. tube.

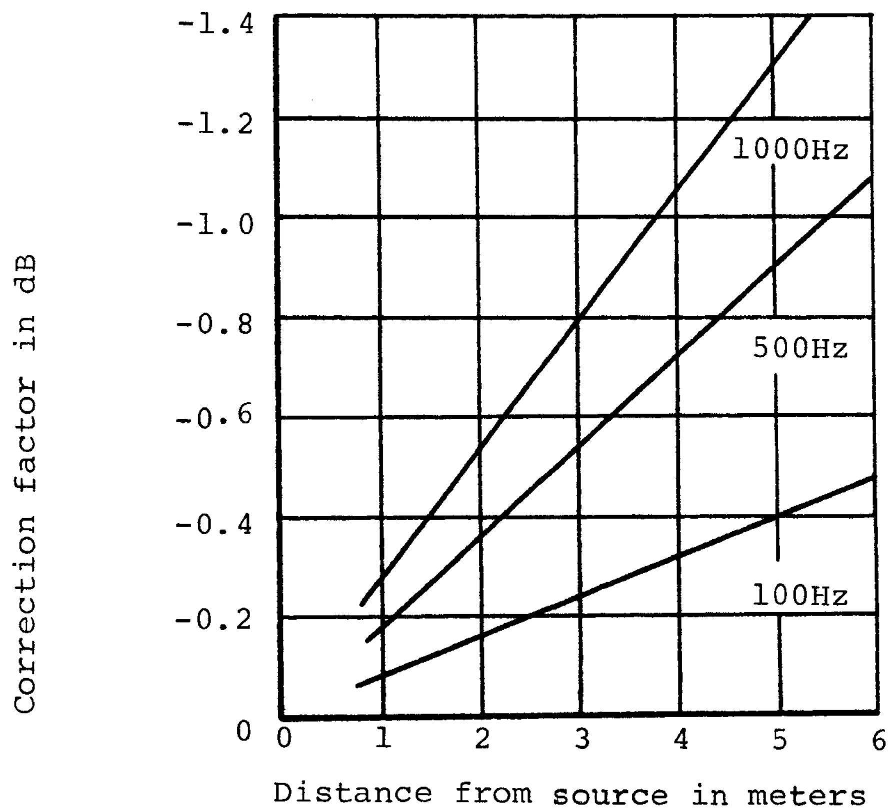

Fig. 6-2. Correction factor for data of Fig. 4, for attenuation in 2 in. dia. tube.

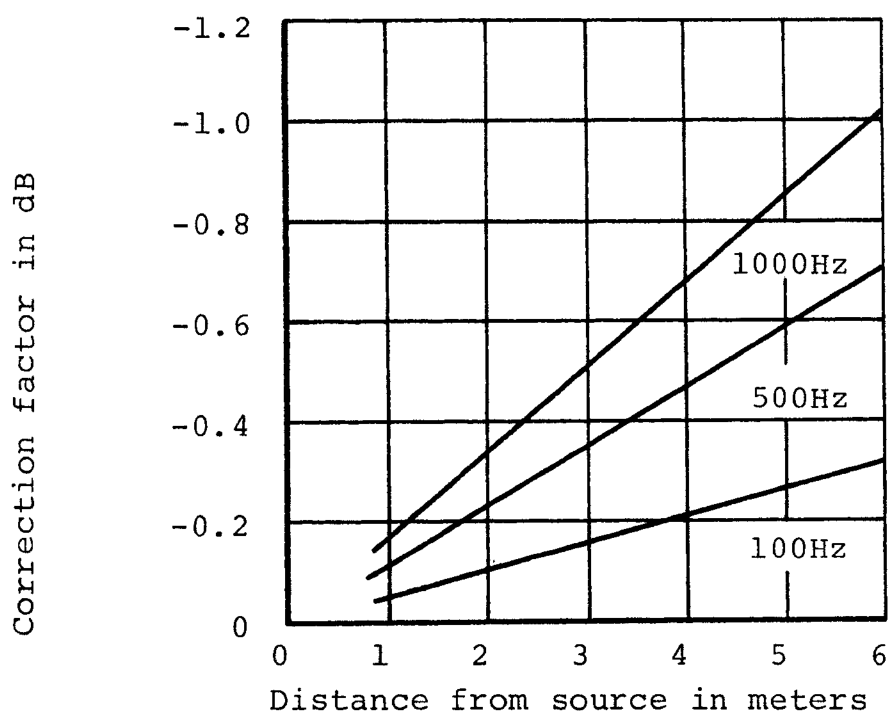

Fig. 6-3. Correction factor for data of Fig. 4, for attenuation in 3 in. dia. tube.

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VII. VITA

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